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S-OPERATIONS IN REPRESENTATION THEORY(1)

BY

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ABSTRACT. For G a group and $\mathbf{A}^G$ the category of G-objects in a category $\mathbf{A}$, a collection of functors, called "S-operations," is introduced under mild restrictions on $\mathbf{A}$. With certain assumptions on $\mathbf{A}$ and with G the symmetric group S_k, one obtains a unigeneration theorem for the Grothendieck ring formed from the isomorphism classes of objects in $\mathbf{A}^{S_k}$. For $\mathbf{A}$ = *finite-dimensional vector spaces over* C, the result says that the representation ring $R(S_k)$ is generated, as a λ-ring, by the canonical k-dimensional permutation representation. When $\mathbf{A}$ = *finite sets*, the S-operations are called "β-operations," and the result says that the Burnside ring $B(S_k)$ is generated by the canonical S_k-set if β-operations are allowed along with addition and multiplication.

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A. Introduction. In the theory of linear representations of a finite group G, representations can be added, multiplied, and formed into a ring $R(G)$, the representation ring of G. In addition, nth symmetric power operations can be applied to any representation, and these operations can be extended to all elements of $R(G)$. Knutson [5] gives a detailed account of these operations in $R(G)$; Atiyah [1] discusses similar operations in the setting of vector bundles.

This paper attempts to generalize these notions. For any group G, a collection of operations on the category $\mathbf{A}^G$ is defined under mild restrictions on $\mathbf{A}$. In the case of linear representations of a finite group, these operations are combinations of symmetric powers, but, in general, they include other operations as well. Letting $G = S_k$ and with certain assumptions on $\mathbf{A}^{S_k}$, one obtains the main result:

COROLLARY II.22. $\langle X_k \rangle = K_0(\mathbf{A}^{S_k})$.

Here, $K_0(\mathbf{A}^{S_k})$ is the Grothendieck ring formed from the isomorphism classes of objects in $\mathbf{A}^{S_k}$, X_k is a particular object in $\mathbf{A}^{S_k}$, and $\langle X_k \rangle$ is the

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subring of $K_0(A^{S_k})$ obtained by applying the operations to X_k and taking sums and products of the results. A principal application of this corollary is that $R(S_k)$ is generated by the canonical permutation representation X_k if symmetric powers are included along with addition and multiplication. For the reader familiar with λ-rings, this statement says that $R(S_k)$ is generated by one element as a λ-ring [2], [5].

§§I.B and I.C present some background on the two principal examples, the Burnside and representation rings of a finite group. Chapter II introduces the S-operations and explores their behavior; the main theorem and its Corollary II.22 are proved in §C. In Chapter III, Corollary II.22 is used to prove that $R(S_k) = \langle X_k \rangle$ and $B(S_k) = \langle X_k \rangle$ for all $k \geqslant 1$. It is also shown that, in general, neither $B(S_k)$ nor $R(S_k)$ is unigenerated as a ring. Moreover, although $B(S_k) = \langle X_k \rangle$, if one allows only symmetric power operations rather than all S-operations, one does not necessarily obtain all of $B(S_k)$.

B. **The Burnside ring,** $B(G)$. Let G be a finite group. A G-set is a finite set T together with a mapping $G \times T \longrightarrow T$ such that $(g_1g_2)t = g_1(g_2t)$, $1t = t$, for all $g_1, g_2 \in G$, $t \in T$. A morphism of G-sets, or G-map, is a set map $f\colon T \longrightarrow T'$, with T and T' G-sets, such that $f(gt) = gf(t)$ for all $g \in G$, $t \in T$. Two G-sets are said to be isomorphic if there is a G-map between them which is a set isomorphism. G-sets and G-maps clearly form a category.

EXAMPLES I.1. (i) Let G be any finite group, T any finite set. Then T can be given the trivial action $gt = t$ for all $g \in G$, $t \in T$.

(i′) In example (i), if T has only one element, T is denoted by 1_G. (Of course, all one-element G-sets are isomorphic.)

(ii) Let H be a subgroup of a finite group G. Then G/H, the set of left cosets of H in G, is a G-set by the action $g(xH) = (gx)H$.

(iii) Let S_n be the symmetric group on the symbols $1, 2, \cdots, n$. Let X_n be the set $\{x_1, x_2, \cdots, x_n\}$, and let S_n act on X_n by $\sigma x_i = x_{\sigma(i)}$. X_n will be called the canonical S_n-set.

(iv) Let G be any finite group. The empty set $\varnothing$ is clearly a G-set.

If T_1 and T_2 are G-sets, then the disjoint union $T_1 \amalg T_2$ is a G-set, under the obvious action. On the other hand, every G-set can be decomposed into its G-orbits:

PROPOSITION I.2. *Every G-set $T \neq \varnothing$ is of the form $\coprod_{i=1}^n T_i$, where T_i is a transitive G-set. The T_i's are unique up to order. (A G-set T is transitive if $T \neq \varnothing$ and if given $t_1, t_2 \in T$ there is a $g \in G$ such that $gt_1 = t_2$.)*

PROPOSITION I.3. *If H is a subgroup of G, then G/H is a transitive*

G-set. Conversely, every transitive G-set is of the form G/H for some subgroup H of G.

PROOF. Given $g_1H, g_2H \in G/H$, $(g_1g_2^{-1})g_2H = g_1H$. Hence G/H is a transitive G-set.

Suppose T is a transitive G-set. Let $t \in T$. Then $T = Gt$. Let G_t be the isotropy group of t, i.e., $G_t = \{g \in G | gt = t\}$. Then the map $T \longrightarrow G/G_t$ defined by $gt \longmapsto gG_t$ is a G-isomorphism.□

PROPOSITION I.4. *$G/H \cong G/K$ as G-sets if and only if H and K are conjugate subgroups of G.*

PROOF. Suppose H and K are conjugate, i.e., $K = g_1^{-1}Hg_1$ for some $g_1 \in G$. Then the maps

$$\phi\colon\ G/H \longrightarrow G/K, \qquad \psi\colon\ G/K \longrightarrow G/H,$$

$$gH \longmapsto (gg_1)K, \qquad gK \longmapsto (gg_1^{-1})H,$$

are G-maps, and $\phi \circ \psi = 1_{G/K}$, $\psi \circ \phi = 1_{G/H}$. So $G/H \cong G/K$.

Conversely, assume $G/H \cong G/K$. Then there exist G-maps $\phi\colon G/H \longrightarrow G/K$, $\psi\colon G/K \longrightarrow G/H$ such that $\phi \circ \psi = 1_{G/K}$, $\psi \circ \phi = 1_{G/H}$. If $\phi(1H) = g_1K$, then $g_1K = hg_1K$ for all $h \in H$, so $g_1^{-1}Hg_1 \subset K$. Similarly $\phi(1K) = g_2H$ gives $g_2^{-1}Kg_2 \subset H$. Thus $g_2^{-1}g_1^{-1}Hg_1g_2 \subset g_2^{-1}Kg_2 \subset H$. Since $g_2^{-1}g_1^{-1}Hg_1g_2$ has the same number of elements as H, $g_2^{-1}g_1^{-1}Hg_1g_2 = g_2^{-1}Kg_2 = H$.□

If T_1 and T_2 are G-sets, then the cartesian product $T_1 \times T_2$ is a G-set under the obvious action. The Burnside ring of G, $B(G)$, consists of all finite formal sums, $\Sigma_i n_i [T_i]$ $(n_i \in Z)$, of G-sets T_i, modulo the relations

(i) $[T_1] = [T_2]$ if $T_1 \cong T_2$ as G-sets,

(ii) $[T_1 \amalg T_2] = [T_1] + [T_2]$.

$B(G)$ is clearly an abelian group; the cartesian product, together with 1_G, gives $B(G)$ the structure of a commutative ring with identity, i.e., $[T_1][T_2] = [T_1 \times T_2]$. Whenever no confusion could arise, the brackets will be omitted.

Propositions I.2, I.3, and I.4 imply

PROPOSITION I.5. *Let $\{H_\alpha\}$ be a set of representatives of the conjugacy classes of subgroups of G. Then $B(G)$ is a free Z-module with basis $\{[G/H_\alpha]\}$.*

The rest of this section is devoted to defining a set map $h_n\colon B(G) \longrightarrow B(G)$ for each integer $n \geqslant 0$. For any G-set T, the set T_G is defined to be the collection of elements of T with the identification $t_1 \sim t_2$ iff $Gt_1 = Gt_2$.

Let T be a G-set. Then $T^n = T \times T \times \cdots \times T$ (n times) is a G-set and also an S_n-set via $\sigma(t_1, \cdots, t_n) = (t_{\sigma^{-1}(1)}, \cdots, t_{\sigma^{-1}(n)})$ for $\sigma \in S_n$. For each integer $n \geqslant 1$, let $h_n(T)$ denote $(T^n)_{S_n}$; $h_n(T)$ is thus the nth symmetric power of T. Since the G- and S_n- actions on T^n commute, $h_n(T) = (T^n)_{S_n}$ is actually a G-set. Clearly h_n sends isomorphic G-sets to isomorphic G-sets. Finally, define $h_0(T)$ to be 1_G for all G-sets T.

If T_1, T_2 are G-sets, then

$$h_n(T_1 \amalg T_2) = \coprod_{i=0}^{n} (h_i(T_1) \times h_{n-i}(T_2)).$$

h_n can now be defined on any element of $B(G)$ by the following construction:

Define

$$H_n\colon \ (G\text{-sets}) \times (G\text{-sets}) \longrightarrow B(G)$$

inductively by

$$H_0(T_1, T_2) = 1_G,$$

$$H_n(T_1, T_2) = h_n(T_1) - \sum_{i=1}^{n-1} H_i(T_1, T_2) h_{n-i}(T_2) \quad \text{for } n > 0.$$

Clearly,

$$T_1 \cong U_1, \ T_2 \cong U_2 \Rightarrow H_n(T_1, T_2) = H_n(U_1, U_2) \quad \text{for all } n \geqslant 0.$$

In addition, an induction argument and the "addition formula" above give

$$H_n(T_1 \amalg T, \ T_2 \amalg T) = H_n(T_1, T_2), \qquad H_n(T, \varnothing) = h_n(T)$$

for all $n \geqslant 0$ and G-sets T_1, T_2, T.

An arbitrary element of $B(G)$ looks like $T_1 - T_2$, where T_1 and T_2 are G-sets. If $T_1 - T_2 = U_1 - U_2$, then $T_1 \amalg U_2 \cong U_1 \amalg T_2$, so

$$H_n(T_1, T_2) = H_n(T_1 \amalg U_2, \ T_2 \amalg U_2)$$

$$= H_n(U_1 \amalg T_2, \ T_2 \amalg U_2) = H_n(U_1, U_2).$$

Thus $H_n(T_1, T_2)$ depends only on $T_1 - T_2$. Therefore, define $h_n(T_1 - T_2) = H_n(T_1, T_2)$. Then $h_n\colon B(G) \longrightarrow B(G)$ is a well-defined set map and coincides with its former definition if $T \in B(G)$ is actually a G-set.

C. **The representation ring,** $R(G)$. Let G be a finite group. A (linear) representation of G (over C) is a finite-dimensional vector space V over C, together with a group homomorphism $\rho\colon G \longrightarrow \operatorname{Aut} V$. V is called a G-module, and ρ gives an action of G on V. One usually writes

$$V \xrightarrow{g} V, \qquad v \longmapsto gv$$

instead of

$$V \xrightarrow{\rho(g)} V, \qquad v \longmapsto \rho(g)v.$$

A G-module map is a linear transformation $f\colon V \longrightarrow V'$, with V and V' G-modules, such that $f(gv) = gf(v)$ for all $g \in G$, $v \in V$. Two G-modules are said to be isomorphic if there exists a G-module map between them which is also a vector space isomorphism. G-modules and G-module maps clearly form a category.

EXAMPLES I.6. (i) Let G be a finite group, V a finite-dimensional vector space. V can be given the trivial action $gv = v$ for all $g \in G$, $v \in V$.

(i′) A special case of example (i) is $V = 0$.

(i″) In example (i), if $\dim V = 1$, V is denoted by 1_G.

(ii) Let $G = S_n$. Let V have basis $\{v_1, \cdots, v_n\}$, and let S_n act by $\sigma v_i = v_{\sigma(i)}$ for $\sigma \in S_n$. This representation V will be called the canonical S_n-module, and denoted X_n.

(ii′) More generally, suppose $\rho\colon G \longrightarrow S_n$ is a group homomorphism. (ρ is called a permutation representation.) By composing this homomorphism with the one in example (ii), one obtains a linear representation of G, $G \xrightarrow{\rho} S_n \longrightarrow \operatorname{Aut} X_n$. Since a G-set T consisting of n elements is a group homomorphism $G \longrightarrow S_n$, the concept of G-set is the same as the concept of permutation representation of G.

A G-module V is reducible if $V = 0$ or if there is a subspace W of V such that $GW \subset W$, with $W \neq 0$ and $W \neq V$. If V is not reducible, it is called irreducible.

If V_1, V_2 are G-modules, then the vector space coproduct $V_1 \amalg V_2$ is a G-module via the obvious action. A G-module V is said to be decomposable if $V \cong V_1 \amalg V_2$ as a G-module, where $V_i \neq 0$. Propositions I.7–I.9 can be found in any book on group representation theory (see [5], [8]).

PROPOSITION I.7 (MASCHKE). *If $V \neq 0$ is reducible, then V is decomposable.*

PROPOSITION I.8. *Every G-module $V \neq 0$ can be expressed as a finite coproduct $V = \coprod_{i=1}^n V_i$, where each V_i is an irreducible G-module. The V_i's are unique (up to order).*

PROPOSITION I.9. *The number of irreducible representations of G is equal to the number of conjugacy classes of G.*

For G-modules V_1, V_2, $V_1 \otimes V_2$ is a G-module via $g(v_1 \otimes v_2) = gv_1 \otimes gv_2$. The representation ring of G, $R(G)$, consists of all finite formal sums $\Sigma_i n_i [V_i]$ $(n_i \in Z)$, of G-modules V_i, modulo the relations

(i) $[V_1] = [V_2]$ if $V_1 \cong V_2$ as G-modules,

(ii) $[V_1 \amalg V_2] = [V_1] + [V_2]$.

$R(G)$ is clearly an abelian group; the tensor product, together with 1_G, gives

$R(G)$ the structure of a commutative ring with identity, that is $[V_1][V_2] = [V_1 \otimes V_2]$. The brackets will usually be omitted.

Propositions I.8 and I.9 imply

PROPOSITION I.10. *Let* Irrep G = *the set of isomorphism classes of irreducible* G*-modules. Then* $R(G)$ *is a free* Z*-module with basis* $\{[V] \mid V \in$ Irrep $G\}$. *The rank of* $R(G)$ = *the number of conjugacy classes of* G.

As in the case of $B(G)$, symmetric power operations h_n: $R(G) \longrightarrow R(G)$ can be introduced. For any G-module V, define the vector space V_G to be V/W, where W is the subspace of V generated by $\{v - gv \mid v \in V, g \in G\}$. The vector spaces V_G and V^G, where V^G is the subspace of V fixed by G, are seen to be isomorphic by the fact that the linear transformation Y: $V \longrightarrow V$ defined by

$$Y(v) = \frac{1}{|G|} \sum_{g \in G} gV$$

has image V^G and kernel W. In the case of sets, however, the corresponding objects T_G and T^G are not generally isomorphic.

For any G-module V, $V^{\otimes n} = V \otimes \cdots \otimes V$ (n times) is a G-module and also an S_n-module via $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ for $\sigma \in S_n$. For each positive integer n, let $h_n(V)$ denote $(V^{\otimes n})_{S_n}$; $h_n(V)$ is thus the nth symmetric power of V. Since the G- and S_n- actions on $V^{\otimes n}$ commute, $h_n(V) = (V^{\otimes n})_{S_n}$ is a G-module. Define $h_0(V)$ to be 1_G for all G-modules V. Clearly h_n sends isomorphic G-modules to isomorphic G-modules.

For G-modules V_1 and V_2,

$$h_n(V_1 \amalg V_2) = \coprod_{i=0}^{n} (h_i(V_1) \otimes h_{n-i}(V_2)).$$

As in the G-set case, h_n can be defined on any element $V_1 - V_2$ of $R(G)$ by defining H_n: (G-modules) $\times$ (G-modules) $\longrightarrow R(G)$ inductively by

$$H_0(V_1, V_2) = 1_G,$$

$$H_n(V_1, V_2) = h_n(V_1) - \sum_{i=0}^{n-1} H_i(V_1, V_2) h_{n-i}(V_2) \quad \text{for } n > 0,$$

and then using the "addition formula" above to show that $H_n(V_1, V_2)$ depends only on $V_1 - V_2$.

G-sets and *G-modules* are examples of the category discussed in Chapter II. There, a family of functors, called S-operations, is defined. In the case of

G-modules, these S-operations turn out to be sums and products of symmetric powers h_n. In fact, by applying these operations to the canonical S_k-module X_k, one can obtain every element in $R(S_k)$ (see III, §A).

In the case of G-sets, however, the S-operations include more than symmetric powers. In III, §B, one sees that applying sums and products of symmetric power operations h_n to the canonical S_k-set X_k does not always give all of $B(S_k)$, whereas applying all the S-operations to X_k does.

II. S-OPERATIONS

A. **The category $\mathbf{A}^{\mathbf{G}}$ and functors ϕ_{W_n}: $\mathbf{A}^{\mathbf{G}} \longrightarrow \mathbf{A}^{\mathbf{G}}$.** Let G be a group, and $\mathbf{A}$ a category. A G-object in $\mathbf{A}$ is an object A in $\mathbf{A}$, together with morphisms $A \xrightarrow{\rho_g} A$ for all $g \in G$, satisfying $\rho_{gh} = \rho_g \circ \rho_h, \rho_1 = 1_A$. $A \xrightarrow{\rho_g} A$ is usually written $A \xrightarrow{g} A$.

A G-map, or G-morphism, is a morphism f: $A \longrightarrow B$ in $\mathbf{A}$, with A and B G-objects, such that $fg = gf$ for all $g \in G$. The category of G-objects and G-maps in $\mathbf{A}$ is denoted $\mathbf{A}^{\mathbf{G}}$.

The aim of this section is to define a collection of functors from $\mathbf{A}^{\mathbf{G}}$ to $\mathbf{A}^{\mathbf{G}}$, under certain assumptions on $\mathbf{A}$. The reader is referred to [6] for a reference on category theory.

Recall that given two morphisms α, β: $A \longrightarrow B$, μ: $B \longrightarrow K$ is a coequalizer for α and β if $\mu\alpha = \mu\beta$, and if whenever μ': $B \longrightarrow K'$ satisfies $\mu'\alpha = \mu'\beta$, then there is a unique morphism γ: $K \longrightarrow K'$ such that $\gamma\mu = \mu'$. Given two morphisms f_1: $A \longrightarrow B_1$, f_2: $A \longrightarrow B_2$, a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & B_2 \arrow[d, "\mu_2"] \\ B_1 \arrow[r, "\mu_1"'] & P \end{tikzcd}$$

is called a pushout for f_1 and f_2 if for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & B_2 \arrow[d, "\mu_2'"] \\ B_1 \arrow[r, "\mu_1'"'] & P' \end{tikzcd}$$

there is a unique morphism γ: $P \longrightarrow P'$ such that $\mu_1' = \gamma u_1$ and $\mu_2' = \gamma u_2$.

LEMMA II.1. *Let* $\mathbf{A}$ *be a category with coequalizers and finite coproducts. Then* $\mathbf{A}$ *has pushouts.*

PROOF. Consider

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & B_2 \\ B_1 & \end{tikzcd}$$

The coproduct $B_1 \amalg B_2$, together with the canonical morphisms i_j: $B_j \longrightarrow B_1 \amalg B_2$, $j = 1, 2$, gives morphisms $i_j \circ f_j$: $A \longrightarrow B_1 \amalg B_2$, $j = 1, 2$. Let μ: $B_1 \amalg B_2 \longrightarrow K$ be the coequalizer for $i_1 \circ f_1$ and $i_2 \circ f_2$. Then $\mu \circ (i_1 \circ f_1) = \mu \circ (i_2 \circ f_2)$ gives a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & B_2 \arrow[d, "\mu \circ i_2"] \\ B_1 \arrow[r, "\mu \circ i_1"'] & K \end{tikzcd}$$

The fact that this diagram is actually a pushout follows from the definitions of coproduct and coequalizer. □

Given a family $\{\mu_i: A \longrightarrow A_i\}_{i \in I}$ of epimorphisms, μ: $A \longrightarrow A'$ is the cointersection of the family if for each $i \in I$ there exist morphisms ν_i: $A_i \longrightarrow A'$ such that $\mu = \nu_i \mu_i$, and if every morphism $A \longrightarrow B$ which factors through each μ_i factors uniquely through μ.

LEMMA II.2. *If* $\mathbf{A}$ *has pushouts, then* $\mathbf{A}$ *has finite cointersections.*

PROOF. It suffices to show existence for a family of two epimorphisms μ_1: $A \longrightarrow A_1$, μ_2: $A \longrightarrow A_2$. Let

$$\begin{tikzcd} A \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & A_2 \arrow[d, "\nu_2"] \\ A_1 \arrow[r, "\nu_1"'] & P \end{tikzcd}$$

be the pushout for μ_1 and μ_2. Then $\nu_1 \mu_1 = \nu_2 \mu_2$: $A \longrightarrow P$ is the cointersection of μ_1 and μ_2 by the definition of pushout. □

Let $\mathfrak{F}_G$: $\mathbf{A} \longrightarrow \mathbf{A}^{\mathbf{G}}$ be the functor which sends $A \in \mathbf{A}$ to $A \in \mathbf{A}^{\mathbf{G}}$ by letting $A \xrightarrow{g} A$ be $A \xrightarrow{1_A} A$ for all $g \in G$. Let $V \in \mathbf{A}^{\mathbf{G}}$. A G-orbit space of V is a pair (O, π), where $O \in \mathbf{A}$ and $\pi \in \mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(V, \mathfrak{F}_G(O))$, such that whenever $X \in \mathbf{A}$ and $f \in \mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(V, \mathfrak{F}_G(X))$ there is a unique $\phi \in \mathrm{Mor}_{\mathbf{A}}(O, X)$ such that $\mathfrak{F}_G(\phi) \circ \pi = f$. When such an O exists, it is of course unique up to natural isomorphism and is denoted V_G.

PROPOSITION II.3. *Let G be a finite group and let $\mathbf{A}$ have coequalizers and finite coproducts. Then (V_G, π) exists for all $V \in \mathbf{A}^{\mathbf{G}}$.*

PROOF. For each pair of distinct elements $g_i, g_j \in G$, let μ_{g_i,g_j}: $V \longrightarrow K_{g_i,g_j}$ be a coequalizer for the morphisms g_i, g_j: $V \longrightarrow V$. Each μ_{g_i,g_j} is an epimorphism since every coequalizer is. Let π: $V \longrightarrow O$ be the cointersection (exists by Lemmas II.1, II.2) of the finite family $\{\mu_{g_i,g_j}: V \longrightarrow K_{g_i,g_j}\}_{g_i \neq g_j}$ in $\mathbf{A}$. This construction gives (V_G, π):

For each $g \in G$, $\mu_{g,1}g = \mu_{g,1}1_V = \mu_{g,1}$, so $\pi g = \pi$ for all $g \in G$. Hence $\pi \in \mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(V, F_G(O))$. If $f \in \mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(V, F_G(X))$, then $fg_i = fg_j$ for all g_i, $g_j \in G$, so f factors through each μ_{g_i,g_j}. Thus there is a unique $\phi \in \mathrm{Mor}_A(O, X)$ such that $\phi \circ \pi = f$.□

REMARK II.4. Proposition II.3 says there is a functor $(\ )_G$: $\mathbf{A}^{\mathbf{G}} \longrightarrow \mathbf{A}$ left adjoint to F_G: $\mathbf{A} \longrightarrow \mathbf{A}^{\mathbf{G}}$, i.e., $\mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(V, F_G(X)) \approx \mathrm{Mor}_{\mathbf{A}}(V_G, X)$, natural in arguments V and X.

For each integer $n \geqslant 1$ and each $W_n \in \mathbf{A}^{\mathbf{S_n}}$ (S_n is the symmetric group), a functor ϕ_{W_n}: $\mathbf{A}^{\mathbf{G}} \longrightarrow \mathbf{A}^{\mathbf{G}}$ will be defined. To do so, assume that $\mathbf{A}$ has not only coequalizers and finite coproducts but also a "tensor product" $\perp$, that is, a functor $\perp$: $\mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}$ which is coherently associative and commutative (see [7, Chapter I]), and which distributes with the coproduct. This insures natural isomorphisms

$$(A_1 \perp A_2) \perp A_3 \approx A_1 \perp (A_2 \perp A_3),$$

$$A_1 \perp A_2 \approx A_2 \perp A_1,$$

$$A_1 \perp (A_2 \amalg A_3) \approx (A_1 \perp A_2) \amalg (A_1 \perp A_3),$$

such that isomorphisms between products of several factors, obtained by successively applying the above, are the same.

Fix $W_n \in \mathbf{A}^{\mathbf{S_n}}$. Let $T \in \mathbf{A}$ and let $T^{\perp n} = T \perp T \perp \cdots \perp T$ (n times). $T^{\perp n} \in \mathbf{A}^{\mathbf{S_n}}$ via the natural isomorphisms which permute its factors. Since $\perp$ is a functor $\mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}$ it induces a functor $\perp$: $\mathbf{A}^{\mathbf{G}} \times \mathbf{A}^{\mathbf{G}} \longrightarrow \mathbf{A}^{\mathbf{G}}$ for any group G; that is if $A \xrightarrow{g} A, B \xrightarrow{g} B$, then $A \perp B \xrightarrow{g \perp g} A \perp B$ gives $A \perp B$ a well-defined G-action by the functoriality of $\perp$. Hence $W_n \perp T^{\perp n} \in \mathbf{A}^{\mathbf{S_n}}$. Defining $\phi_{W_n}(T)$ to be $(W_n \perp T^{\perp n})_{S_n}$, one obtains a functor ϕ_{W_n}: $\mathbf{A} \longrightarrow \mathbf{A}$. (For f: $T \longrightarrow T'$, $\phi_{W_n}(f)$: $(W_n \perp T^n)_{S_n} \longrightarrow (W_n \perp T'^n)_{S_n}$ is the obvious map.)

If $T \in \mathbf{A}^{\mathbf{G}}$, T comes with morphisms $T \xrightarrow{g} T$ for all $g \in G$, which induce morphisms

$$\phi_{W_n}(T) \xrightarrow{\phi_{W_n}(g)} \phi_{W_n}(T)$$

for all $g \in G$. Since ϕ_{W_n} is a functor, the maps $\phi_{W_n}(g)$ define a G-action on $\phi_{W_n}(T)$. Thus one has a functor ϕ_{W_n}: $\mathbf{A}^{\mathbf{G}} \longrightarrow \mathbf{A}^{\mathbf{G}}$ for any group G.

In conclusion, then, if G is any group and if $\mathbf{A}$ has coequalizers, finite coproducts, and a "tensor product" $\perp$, then for each positive integer and each $W_n \in \mathbf{A}^{\mathbf{S_n}}$, one has a functor ϕ_{W_n}: $\mathbf{A}^{\mathbf{G}} \longrightarrow \mathbf{A}^{\mathbf{G}}$ defined by $\phi_{W_n}(T) = (W_n \perp T^{\perp n})_{S_n}$.

B. **The behavior of the functors** ϕ_{W_n}. The purpose of this section is to investigate the behavior of the functors ϕ_{W_n}. To do so, one first introduces induced objects.

Suppose $H \subset G$ are groups. $A \in \mathbf{A}^{\mathbf{G}}$ may be viewed as an H-object via the inclusion $H \hookrightarrow G$, giving rise to a functor Res_H^G: $\mathbf{A}^{\mathbf{G}} \longrightarrow \mathbf{A}^{\mathbf{H}}$. Let $W \in \mathbf{A}^{\mathbf{H}}$. An induced object of W is a pair (V, ψ), where $V \in \mathbf{A}^{\mathbf{G}}$ and $\psi \in \mathrm{Mor}_{\mathbf{A}^{\mathbf{H}}}(W, \mathrm{Res}_{\mathbf{H}}^{\mathbf{G}} V)$ such that whenever $X \in \mathbf{A}^{\mathbf{G}}$ and $f \in \mathrm{Mor}_{\mathbf{A}^{\mathbf{H}}}(W, \mathrm{Res}_{\mathbf{H}}^{\mathbf{G}} X)$ there is a unique $\phi \in \mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(V, X)$ satisfying $(\mathrm{Res}_{\mathbf{H}}^{\mathbf{G}}\phi) \circ \psi = f$. When such a V exists, it is unique up to natural isomorphism and is denoted $\mathrm{Ind}_{\mathbf{H}}^{\mathbf{G}} W$.

PROPOSITION II.5. *Let $H \subset G$ be finite groups and let $\mathbf{A}$ have coequalizers and finite coproducts. Then $(\mathrm{Ind}_{\mathbf{H}}^{\mathbf{G}} W, \psi)$ exists for all $W \in \mathbf{A}^{\mathbf{H}}$.*

PROOF. Let $W \in \mathbf{A}^{\mathbf{H}}$. Form the coproduct of W with itself $|G|$ times to obtain the object $\amalg_{x \in G} W_x$ in $\mathbf{A}$, which is in $\mathbf{A}^{\mathbf{G}}$ via the maps $\amalg W_x \xrightarrow{*g} \amalg W_x$, for all $g \in G$, which permute the factors; more precisely, $*g$ is induced by the maps $*g_x$: $W_x \longrightarrow W_{gx} \hookrightarrow \amalg_{x \in G} W_x$, where the first morphism is 1_W and the second is the canonical map associated with the coproduct. In the future, $W_x \xrightarrow{1_W} W_y$ will be denoted 1_x^y. For $h \in H$, let $\amalg W \xrightarrow{h^*} \amalg W_x$ be the map induced from maps

$$h_x^*\colon\ W_x \xrightarrow{1_x^{xh}} W_{xh} \hookrightarrow \coprod W_x,$$

and let $\amalg W_x \xrightarrow{h} \amalg W_x$ be induced from the maps h_x: $W_x \xrightarrow{h} W_x \hookrightarrow \amalg W_x$, where the first map is just the action of H on W.

Observe that $h^{*}{*g} = {*gh^*}$ and $h({*g}) = {*gh}$ for all $g \in G, h \in H$. For $h \in H$, let μ_h: $\amalg W_x \longrightarrow K_h$ be the coequalizer of h^* and h. Since $_\theta(\mu_h^* g)h^* = \mu_h h^{**}g = (\mu_h h)^*g = (\mu_h^* g)h$ for $g \in G$, there is a unique map $K_h \xrightarrow{g} K_h$ such that the triangle

$$\begin{tikzcd} \coprod W_x \arrow[r, "\mu_h^* g"] \arrow[d, "\mu_h"'] & K_h \\ K_h \arrow[ur, dashed, "\theta_g"'] & \end{tikzcd}$$

commutes. One thus obtains a map θ_g for each $g \in G$. The uniqueness of each

θ_g implies $\theta_1 = 1_{K_h}$ and $\theta_{g_1 g_2} = \theta_{g_1}\theta_{g_2}$. Hence $K_h \in \mathbf{A}^G$ and μ_h is a G-map.

Let μ: $\amalg W_x \longrightarrow K$ be the cointersection (exists by Lemmas II.1, II.2) of the finite family of epimorphisms $\{\mu_h\}_{h\in H}$ in **A**. Since μ factors through each μ_h, μ^*g does also; hence for each $g \in G$, there is a unique map $K \xrightarrow{\rho_g} K$ such that the triangle

$$\begin{tikzcd} \coprod W_x \arrow[r, "\mu^*g"] \arrow[d, "\mu"'] & K \\ K \arrow[ur, dashed, "\rho_g"'] & \end{tikzcd}$$

commutes. The uniqueness of each ρ_g makes K a G-object and therefore μ a G-map.

Let ψ: $W \longrightarrow \operatorname{Res}_H^G K$ be the map μ_1: $W_1 \longrightarrow K$ which comes from μ: $\amalg_{x\in G} W_x \longrightarrow K$. (Here and elsewhere, W is identified with W_1.) $K = \operatorname{Ind}_H^G W$ by the following argument:

To show that ψ is an H-map, one must show that $\rho_h\psi = \psi h$. The last commutative triangle and the definition of μ give $\rho_h\psi = \rho_h\mu_1 = \mu^*h_1 = \mu h_1 = \mu_1 h = \psi h$.

Suppose f: $W \longrightarrow \operatorname{Res}_H^G X$ is an H-map. One can show that there is a unique G-map τ: $\amalg W_x \longrightarrow X$ such that the triangle

$$\begin{tikzcd} W_1 \arrow[r, "f"] \arrow[d, hook] & X \\ \coprod W_x \arrow[ur, dashed, "\tau"'] & \end{tikzcd}$$

commutes, as follows:

If such a τ exists, then for each $x \in G$, the square

$$\begin{tikzcd} W_1 \arrow[r, "f"] \arrow[d, "*x_1"'] & X \arrow[d, "x"] \\ \coprod W_x \arrow[r, "\tau"'] & X \end{tikzcd}$$

commutes. Hence $\tau_x = xf1_x^1$ for all $x \in G$. Thus τ is unique. For existence, define τ by $\tau_x = xf1_x^1$ for all $x \in G$.

Moreover, $\tau h^* = \tau h$ for all $h \in H$ since $(\tau h^*)_x = \tau_{xh}1_x^{xh} = xhf1_{xh}^1 1_x^{xh} = xhf1_x^1$, $(\tau h)_x = \tau_x h = xf1_x^1 h = xfh1_x^1$, and f is an H-map. Hence τ factors through μ_h for all $h \in H$. It follows that there is a unique ϕ: $K \longrightarrow X$ satisfying $\tau = \phi\mu$. By the fact that μ and τ are G-maps and by the uniqueness of ϕ, ϕ is itself a G-map: $(g\phi\rho_{g^{-1}})\mu = g\phi\mu^*g^{-1} = g\tau g^{-1} = gg^{-1}\tau = \tau \Rightarrow g\phi\rho_{g^{-1}} = \phi$, or, $g\phi = \phi\rho_g$.

The commutativity of the two small triangles in the figure

$$\begin{tikzcd} W_1 \arrow[r, "f"] \arrow[d, hook] & X \\ \amalg W_x \arrow[ur, "\tau"] \arrow[d, "\mu"'] & \\ K \arrow[uur, "\phi"'] & \end{tikzcd}$$

implies the commutativity of the large triangle. Hence ϕ is a G-map satisfying $(\mathrm{Res}_H^G\phi) \circ \psi = f$. In addition, if ϕ_1 makes the diagram

$$\begin{tikzcd} W_1 \arrow[r, "f"] \arrow[d, hook] & V \\ \amalg W_x \arrow[d, "\mu"'] & \\ K \arrow[uur, "\phi_1"'] & \end{tikzcd}$$

commute, then $\phi_1\mu = \tau$ by uniqueness of τ, and so $\phi_1 = \phi$ by uniqueness of $\varnothing$.□

REMARK II.6. Proposition II.5 says there is a functor $\mathrm{Ind}_H^G\colon \mathbf{A}^{\mathbf{H}} \longrightarrow \mathbf{A}^{\mathbf{G}}$ left adjoint to $\mathrm{Res}_H^G\colon \mathbf{A}^{\mathbf{G}} \longrightarrow \mathbf{A}^{\mathbf{H}}$, i.e., $\mathrm{Mor}_{\mathbf{A}^{\mathbf{H}}}(W, \mathrm{Res}_H^G V) \approx \mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(\mathrm{Ind}_H^G W, V)$, natural in arguments W and V.

PROPOSITION II.7. *Let G be a group and $\mathbf{A}$ a category with finite coproducts. Suppose $V \in \mathbf{A}^{\mathbf{G}}$ and $V = \amalg_{i=1}^n W_i$ as an object in $\mathbf{A}$. Assume G permutes the W_i's transitively, that is, each $g_i\colon W_i \longrightarrow \amalg_{i=1}^n W_i$ looks like $W_i \longrightarrow W_j \hookrightarrow \amalg_{i=1}^n W_i$ for some j and some morphism $W_i \longrightarrow W_j$, and given any i, j, there is a $g \in G$ such that $g_i\colon W_i \longrightarrow W_j \hookrightarrow \amalg_{i=1}^n W_i$.*

Let W_{i_0} be one of the W_i's and let H be its isotropy group, i.e., $H = \{g \in G | g_{i_0}\colon W_{i_0} \longrightarrow W_{i_0} \hookrightarrow \amalg W_i\}$. Then as an object in $\mathbf{A}^{\mathbf{G}}$, $V = \mathrm{Ind}_H^G W_{i_0}$.

PROOF. If $g_i\colon W_i \longrightarrow W_j \hookrightarrow \amalg W_i$, denote the map $W_i \longrightarrow W_j$ by g_i^j. Since $gg^{-1} = g^{-1}g = 1_v$, $g_i\colon W_i \longrightarrow W_j \hookrightarrow \amalg W_i$ implies $g_j^{-1}\colon W_j \longrightarrow W_i \hookrightarrow \amalg W_i$, and $g_i^j(g^{-1})_j^i = 1_{W_j}$, $(g^{-1})_j^i g_i^j = 1_{W_i}$.

Let $\psi\colon W_{i_0} \longrightarrow \mathrm{Res}_H^G V$ be the canonical map $W_{i_0} \hookrightarrow \amalg_{i=1}^n W_i$. ψ is clearly an H-map. To show $V = \mathrm{Ind}_H^G W_{i_0}$, one need only show that V satisfies the appropriate universal property.

Suppose $f \in \mathrm{Mor}_{\mathbf{A}^{\mathbf{H}}}(W_{i_0}, \mathrm{Res}_H^G X)$. If there is a G-map $\phi\colon V \longrightarrow X$ such that $(\mathrm{Res}_H^G\phi) \circ \psi = f$, then for each $g \in G$ there is a commutative diagram

$$\begin{tikzcd} W_{i_0} \arrow[r, "f"] \arrow[d, "g_{i_0}"'] & X \arrow[d, "g"] \\ \amalg W_i \arrow[r, "\phi"'] & X \end{tikzcd}$$

Given i, let $g \in G$ be such that g_{i_0}: $W_{i_0} \longrightarrow W_i \hookrightarrow \amalg W_i$. Then $\phi_i = \phi g_{i_0}(g^{-1})_i^{i_0} = gf(g^{-1})_i^{i_0}$. Thus such a ϕ is unique.

To show existence, define ϕ by $\phi_i = gf(g^{-1})_i^{i_0}$, where $g \in G$ such that g_{i_0}: $W_{i_0} \longrightarrow W_i \hookrightarrow \amalg W_i$. ϕ is well defined: If $\tilde{g}, g$: $W_{i_0} \longrightarrow W_i \hookrightarrow \amalg W_i$, then $\tilde{g}^{-1}g \in H$, so that $\tilde{g}^{-1}gf = f(\tilde{g}^{-1}g)_{i_0}^{i_0} = f(\tilde{g}^{-1})_i^{i_0} g_{i_0}^i$; hence $gf(g^{-1})_i^{i_0} = \tilde{g}f(\tilde{g}^{-1})_i^{i_0}$.

LEMMA II.8. *Let* **A** *have coequalizers and finite coproducts, and let* G *be a finite group. Then*

$$(A \amalg B)_G \approx A_G \amalg B_G,$$

natural in arguments A *and* B.

PROOF. There is a natural isomorphism

$$\mathrm{Mor}_{\mathbf{A}}((A \amalg B)_G, X) \approx \mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(A \amalg B, F_G(X))$$

(Remark II.4). Since adjoints are unique, one need only show

$$\mathrm{Mor}_{\mathbf{A}}(A_G \amalg B_G, X) \approx \mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(A \amalg B, F_G(X)).$$

But

$$\begin{aligned} \mathrm{Mor}_{\mathbf{A}}(A_G \amalg B_G, X) &\approx \mathrm{Mor}_{\mathbf{A}}(A_G, X) \times \mathrm{Mor}_{\mathbf{A}}(B_G, X) \\ &\approx \mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(A, F_G(X)) \times \mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(B, F_G(X)) \approx \mathrm{Mor}_{\mathbf{A}^{\mathbf{G}}}(A \amalg B, F_G(X)). \end{aligned}$$

THEOREM II.9. *Let* G *be a group, and let* **A** *have finite coproducts, coequalizers, and a "tensor product"* $\perp$. *Then if* $W_n, W'_n \in \mathbf{A}^{\mathbf{S_n}}$,

$$\phi_{W_n \amalg W'_n}(T) = \phi_{W_n}(T) \amalg \phi_{W'_n}(T)$$

for all $T \in \mathbf{A}^{\mathbf{G}}$.

PROOF.

$$\begin{aligned} \phi_{W_n \amalg W'_n}(T) &= ((W_n \amalg W'_n) \perp T^{\perp n})_{S_n} \\ &\approx ((W_n \perp T^{\perp n}) \amalg (W'_n \perp T^{\perp n}))_{S_n} \\ &\approx (W_n \perp T^{\perp n})_{S_n} \amalg (W'_n \amalg T^{\perp n})_{S_n} \quad \text{(by Lemma II.8)} \\ &= \phi_{W_n}(T) \amalg \phi_{W'_n}(T). \square \end{aligned}$$

LEMMA II.10. *Let* **A** *have finite coproducts and coequalizers, and let*

$K \subset H \subset G$ be finite groups. Let $U \in \mathbf{A}^{\mathbf{K}}$ and $W, W' \in \mathbf{A}^{\mathbf{H}}$. Then

(i) $\mathrm{Ind}_H^G(W \amalg W') \approx \mathrm{Ind}_H^G W \amalg \mathrm{Ind}_H^G W'$,

(ii) $\mathrm{Ind}_H^G(\mathrm{Ind}_K^H U) \approx \mathrm{Ind}_K^G U$,

(iii) $(\mathrm{Ind}_H^G W)_G \approx W_H$.

All the above isomorphisms are natural.

PROOF. (i) The result follows from the adjointness of Res_H^G and Ind_H^G (Remark II.6) and an argument analogous to the one for Lemma II.8.

(ii) This result follows from the uniqueness of adjoints and the obvious fact that $\mathrm{Res}_K^H(\mathrm{Res}_H^G V) \approx \mathrm{Res}_K^G V$.

(iii) The proof, which is similar to the preceding one, uses the adjointness of Res_H^G and Ind_H^G and of F_G and $(\ )_G$, and the fact that $\mathrm{Res}_H^G(F_G(X)) \approx F_H(X)$.□

LEMMA II.11 (FROBENIUS RECIPROCITY). *Let $H \subset G$ be finite groups, and let $\mathbf{A}$ have finite coproducts, coequalizers, and a "tensor product" $\perp$. Assume there is a functor* Hom: $\mathbf{A}^\circ \times \mathbf{A} \longrightarrow \mathbf{A}$ *such that*

$$\mathrm{Mor}_{\mathbf{A}}(A \perp B, C) \approx \mathrm{Mor}_{\mathbf{A}}(A, \mathbf{Hom}(B, C)),$$

natural in A, B, C. Then for $W \in \mathbf{A}^{\mathbf{H}}$, $V \in \mathbf{A}^{\mathbf{G}}$,

$$(\mathrm{Ind}_H^G W) \perp V \approx \mathrm{Ind}_H^G(W \perp \mathrm{Res}_H^G V).$$

PROOF. The functoriality of **Hom**: $\mathbf{A}^\circ \times \mathbf{A} \longrightarrow \mathbf{A}$ induces the functor **Hom**: $(\mathbf{A}^{\mathbf{G}})^\circ \times \mathbf{A}^{\mathbf{G}} \longrightarrow \mathbf{A}^{\mathbf{G}}$, and clearly

$$\mathrm{Res}_H^G \mathbf{Hom}(V, X) \approx \mathbf{Hom}(\mathrm{Res}_H^G V, \mathrm{Res}_H^G X).$$

The lemma now follows from the standard argument using uniqueness of adjoints.□

The following theorem gives a useful simplification for some of the functors ϕ_{W_n} in the special case of the existence of an object $\mathbf{1}$ in $\mathbf{A}$ such that $A \perp \mathbf{1} \approx A$, natural and coherent in the sense of II, §A. The object $F_G(\mathbf{1}) \in \mathbf{A}^{\mathbf{G}}$ will be denoted $\mathbf{1}_G$, or simply $\mathbf{1}$.

THEOREM II.12. *Let G be a group, let $\mathbf{A}$ have an object $\mathbf{1}$ and be as in Lemma* II.11, *and let* Hom *exist. Let $H \subset S_n$, $T \in \mathbf{A}^{\mathbf{G}}$, and $W_n = \mathrm{Ind}_H^{S_n}\mathbf{1}$. Then $\phi_{W_n}(T) = (\mathrm{Res}_H^{S_n}(T^{\perp n}))_H$.*

PROOF.

$$\begin{aligned}
\phi_{W_n}(T) &= ((\mathrm{Ind}_H^{S_n}\mathbf{1}) \perp T^{\perp n})_{S_n} \\
&\approx (\mathrm{Ind}_H^{S_n}(\mathbf{1} \perp \mathrm{Res}_H^{S_n}(T^{\perp n})))_{S_n} \quad \text{(by Lemma II.11)} \\
&\approx (\mathrm{Ind}_H^{S_n}(\mathrm{Res}_H^{S_n}(T^{\perp n})))_{S_n} \approx (\mathrm{Res}_H^{S_n}(T^{\perp n}))_H \quad \text{(by Lemma II.10).}
\end{aligned}$$
□

EXAMPLES II.13. (i) If $W_n = \mathrm{Ind}_1^{S_n} 1$, then $\phi_{W_n}(T) = (\mathrm{Res}_1^{S_n}(T^{\perp n}))_1 = T^{\perp n}$.

(ii) If $W_n = \mathrm{Ind}_{S_n}^{S_n} 1$, then $\phi_{W_n}(T) = (\mathrm{Res}_{S_n}^{S_n}(T^{\perp n}))_{S_n} = (T^{\perp n})_{S_n}$ = the nth symmetric power of T.

If G and H are groups, $A \in \mathbf{A}^{\mathbf{G}}, B \in \mathbf{A}^{\mathbf{H}}$, then the morphisms $A \xrightarrow{g} A$, $B \xrightarrow{h} B$ induce the morphism $A \perp B \xrightarrow{g \perp h} A \perp B$, thereby making $A \perp B \in \mathbf{A}^{\mathbf{G} \times \mathbf{H}}$ ($G \times H$ is the direct product of G and H). In this setting, one has the following lemma:

LEMMA II.14. *Let G and H be finite groups, and let $\mathbf{A}$ have finite coproducts, coequalizers, and a "tensor product" $\perp$. Assume there is a functor* Hom, *as in Lemma* II.11. *If $A \in \mathbf{A}^{\mathbf{G}}, B \in \mathbf{A}^{\mathbf{H}}$, then $(A \perp B)_{G \times H} \approx A_G \perp B_H$.*

PROOF. Because of the uniqueness of adjoints, one need only show

$$\mathrm{Mor}_{\mathbf{A}^{\mathbf{G}\times\mathbf{H}}}(A \perp B,\ \mathcal{F}_{G\times H}(X)) \approx \mathrm{Mor}_{\mathbf{A}}(A_G \perp B_H,\ X).$$

This follows from the following chain of natural isomorphisms, each easily verifiable:

$$\begin{aligned}
&\mathrm{Mor}_{\mathbf{A}^{\mathbf{G}\times\mathbf{H}}}(A \perp B,\ \ {}_{G\times H}(X)) \\
&\qquad \approx \mathrm{Mor}_{(\mathbf{A}^{\mathbf{H}})^{\mathbf{G}}}(\mathcal{F}_H(A) \perp \mathcal{F}_G(B),\ \mathcal{F}_G(\mathcal{F}_H(X))) \\
&\qquad \approx \mathrm{Mor}_{(\mathbf{A}^{\mathbf{H}})^{\mathbf{G}}}(\mathcal{F}_H(A),\ \mathbf{Hom}(\mathcal{F}_G(B),\ \mathcal{F}_G(\mathcal{F}_H(X)))) \\
&\qquad \approx \mathrm{Mor}_{(\mathbf{A}^{\mathbf{H}})^{\mathbf{G}}}(\mathcal{F}_H(A),\ \mathcal{F}_G(\mathbf{Hom}(B,\ \mathcal{F}_H(X)))) \\
&\qquad \approx \mathrm{Mor}_{\mathbf{A}^{\mathbf{H}}}((\mathcal{F}_H(A))_G,\ \mathbf{Hom}(B,\ \mathcal{F}_H(X))) \\
&\qquad \approx \mathrm{Mor}_{\mathbf{A}^{\mathbf{H}}}(\mathcal{F}_H(A_G),\ \mathbf{Hom}(B,\ \mathcal{F}_H(X))) \\
&\qquad \approx \mathrm{Mor}_{\mathbf{A}^{\mathbf{H}}}(\mathcal{F}_H(A_G) \perp B,\ \mathcal{F}_H(X)) \\
&\qquad \approx \mathrm{Mor}_{\mathbf{A}^{\mathbf{H}}}(B \perp \mathcal{F}_H(A_G),\ \mathcal{F}_H(X)) \\
&\qquad \approx \mathrm{Mor}_{\mathbf{A}^{\mathbf{H}}}(B\ \ \mathbf{Hom}(\mathcal{F}_H(A_G),\ \mathcal{F}_H(X))) \\
&\qquad \approx \mathrm{Mor}_{\mathbf{A}^{\mathbf{H}}}(B,\ \mathcal{F}_H(\mathbf{Hom}(A_G,\ X))) \approx \mathrm{Mor}_{\mathbf{A}}(B_H,\ \mathbf{Hom}(A_G,\ X)) \\
&\qquad \approx \mathrm{Mor}_{\mathbf{A}}(B_H \perp A_G,\ X) \approx \mathrm{Mor}_{\mathbf{A}}(A_G \perp B_H,\ X). \square
\end{aligned}$$

In the next theorem, $S_n \times S_m$ is viewed as a subgroup of S_{n+m} by viewing S_n as permuting the symbols $1, 2, \cdots, n$, S_m the symbols $n+1, n+2, \cdots, n+m$, and S_{n+m} the symbols $1, 2, \cdots, n+m$.

THEOREM II.15. *Let G be a group, let $\mathbf{A}$ have finite coproducts, coequal-*

izers, a "tensor product" $\perp$, *and an object* $\mathbf{1}$. *Assume there is a functor* **Hom** *as in Lemma* II.11. *Let* $W_n \in \mathbf{A}^{S_n}$, $W_m \in \mathbf{A}^{S_m}$, *and* $T \in \mathbf{A}^G$. *If* $W_{n+m} = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} W_n \perp W_m$, *then* $\phi_{W_{n+m}}(T) = \phi_{W_n}(T) \perp \phi_{W_m}(T)$.

PROOF.

$$\phi_{W_{n+m}}(T) = ((\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} W_n \perp W_m) \perp T^{\perp n+m})_{S_{n+m}}$$

$$\approx (\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (W_n \perp W_m \perp \mathrm{Res}_{S_n \times S_m}^{S_{n+m}} T^{\perp n+m}))_{S_{n+m}} \quad \text{(by Lemma II.11)}$$

$$\approx (W_n \perp W_m \perp \mathrm{Res}_{S_n \times S_m}^{S_{n+m}} T^{\perp n+m})_{S_n \times S_m} \quad \text{(by Lemma II.10)}$$

$$\approx ((W_n \perp T^{\perp n}) \perp (W_m \perp T^{\perp m}))_{S_n \times S_m}$$

$$\approx (W_n \perp T^{\perp n})_{S_n} \perp (W_m \perp T^{\perp m})_{S_m} \quad \text{(by Lemma II.14)}$$

$$= \phi_{W_n}(T) \perp \phi_{W_m}(T). \square$$

C. The main theorem and corollary. Let G be a group and $\mathbf{A}$ have finite coproducts, a "tensor product" $\perp$, and an object $\mathbf{1}$. Define the Grothendieck ring $K_0(\mathbf{A}^G)$ to consist of all finite formal sums $\Sigma_i n_i [T_i]$ $(n_i \in \mathbf{Z})$ of G-objects T_i in $\mathbf{A}$, modulo the relations

(i) $[T_1] = [T_2]$ if $T_1 \cong T_2$ as G-objects,

(ii) $[T_1 \amalg T_2] = [T_1] + [T_2]$.

Clearly, $K_0(\mathbf{A}^G)$ is an abelian group; the "tensor product" $\perp$, together with the object $\mathbf{1} \in \mathbf{A}^G$, gives $K_0(\mathbf{A}^G)$ the structure of a commutative ring with identity, i.e. $[T_1][T_2] = [T_1 \perp T_2]$. When the meaning is clear, brackets will be omitted, e.g., $[T_1] - [T_2]$ will appear as $T_1 - T_2$.

EXAMPLES II.16. (i) Let G be a finite group and $\mathbf{A}$ the category of finite sets. Then $\mathbf{A}^G = G$*-sets.* Let $\perp$ be the cartesian product, and $\mathbf{1}$ be any one-element G-set. Then $K_0(\mathbf{A}^G)$ is the Burnside ring of G, $B(G)$. (See I, §B.)

(ii) Let G be a finite group and $\mathbf{A}$ the category of finite-dimensional vector spaces over $\mathbf{C}$. Then $\mathbf{A}^G = G$*-modules.* Let $\perp$ be the tensor product $\otimes$, and $\mathbf{1}$ be the one-dimensional G-module with trivial G-action. Then $K_0(\mathbf{A}^G)$ is the representation ring of G, $R(G)$ (see I, §C).

REMARK II.17. In the above examples, $[T_1] = [T_2]$ implies $T_1 \cong T_2$ as G-objects (see I, §B, §C). This is not the case in general; in particular, if $\mathbf{A}$ is the category of vector bundles over a space X, then $[E] = [F]$ implies only that $E \oplus \mathbf{n} \cong F \oplus \mathbf{n}$, where $\mathbf{n}$ is the trivial bundle of dimension n [3, Appendix].

Let $H \subset G$ be finite groups. Let $\mathbf{A}$ have finite coproducts, coequalizers, a "tensor product" $\perp$, and an object $\mathbf{1}$. $P(\mathbf{A}^G)$ is defined to be the subring of $K_0(\mathbf{A}^G)$ generated by $\{\mathrm{Ind}_H^G \mathbf{1} \mid H$ a subgroup of $G\}$.

PROPOSITION II.18. *Let $H \subset G$ be finite groups. If $W = \mathbf{1}_H$, then* $\mathrm{Ind}_H^G W = \amalg_{\bar{x}\in G/H} \mathbf{1}_{\bar{x}}$ *with G-action given by*

$$g\colon \mathbf{1}_{\bar{x}} \xrightarrow{\mathbf{1}_1} \mathbf{1}_{\overline{gx}} \hookrightarrow \amalg \mathbf{1}_{\bar{x}}.$$

PROOF. Let $\psi\colon \mathbf{1}_{\bar{1}} \hookrightarrow \amalg \mathbf{1}_{\bar{x}}$. One need only show that $(\amalg \mathbf{1}_{\bar{x}}, \psi)$ satisfies the appropriate universal property. Clearly, $\psi\colon \mathbf{1}_{\bar{1}} \longrightarrow \mathrm{Res}_H^G(\amalg \mathbf{1}_{\bar{x}})$ is an H-map. If $f \in \mathrm{Mor}_{\mathbf{A}^H}(\mathbf{1}_H, \mathrm{Res}_H^G X)$, there is a unique $\phi \in \mathrm{Mor}_{\mathbf{A}^G}(\amalg \mathbf{1}_{\bar{x}}, X)$ such that $(\mathrm{Res}_H^G \phi) \circ \psi = f$, namely $\phi_{\bar{x}} x f \mathbf{1}_{\bar{x}}^{\bar{1}}$ for all $\bar{x} \in G/H$.□

PROPOSITION II.19. *Every element in $P(\mathbf{A}^G)$ is of the form $\Sigma_i n_i \mathrm{Ind}_{H_i}^G \mathbf{1}$, where $n_i \in \mathbf{Z}$ and H_i is a subgroup of G.*

PROOF.

$$\begin{aligned}
(\mathrm{Ind}_H^G \mathbf{1}) \perp (\mathrm{Ind}_K^G \mathbf{1}) &\approx \Big(\coprod_{\bar{x}\in G/H} \mathbf{1}_{\bar{x}}\Big) \perp \Big(\coprod_{\bar{y}\in G/K} \mathbf{1}_{\bar{y}}\Big) \quad \text{(by Proposition II.18)}\\
&\approx \coprod_{\bar{x},\bar{y}} (\mathbf{1}_{\bar{x}} \perp \mathbf{1}_{\bar{y}})\\
&\approx \coprod_{\bar{x},\bar{y}} \mathbf{1}_{(\bar{x},\bar{y})} \approx \coprod_{\substack{G\text{-orbits }\alpha\\ \text{of } G/H\times G/K}} \Big(\coprod_{(\bar{x},\bar{y})\in\alpha} \mathbf{1}_{(\bar{x},\bar{y})}\Big)\\
&\approx \coprod_{G\text{-orbits }\alpha} \mathrm{Ind}_{H_\alpha}^G \mathbf{1} \quad \text{(by Proposition II.7).}\square
\end{aligned}$$

The canonical S_k-object in $\mathbf{A}$, denoted X_k, is defined to be $\mathrm{Ind}_{S_1\times S_{k-1}}^{S_k} \mathbf{1}$.
REMARK II.20. From Proposition II.18, it follows that

$$X_k = \coprod_{\bar{\sigma}\in S_k/(S_1\times S_{k-1})} \mathbf{1}_{\bar{\sigma}}.$$

Since $\bar{\sigma} = \bar{\tau}$ in $S_k/(S_1 \times S_{k-1})$ iff $\tau(1) = \sigma(1)$, each $S_1 \times S_{k-1}$-orbit of S_k consists of precisely those $\sigma \in S_k$ which send 1 to the same symbol j. Hence $X_k = \amalg_{j=1}^k \mathbf{1}_j$, where $\sigma \in S_k$ acts by

$$\mathbf{1}_j \xrightarrow{\mathbf{1}_1} \mathbf{1}_{\sigma(j)} \hookrightarrow \amalg \mathbf{1}_j.$$

For examples, see I.1(iii) and I.6(ii).

Let G be a group, and $\mathbf{A}$ have finite coproducts, coequalizers, a "tensor product" $\perp$, and object $\mathbf{1}$. Let $\phi_0\colon \mathbf{A}^G \longrightarrow \mathbf{A}^G$ be the functor sending $A \in$

A^{G} to 1_G. ϕ_0 and the functors ϕ_{W_n} arising from all positive integers n and all $W_n \in \mathrm{A}^{\mathrm{S_n}}$ (see II, §A) will be called S-operations. If $\mathrm{A} =$ *G-modules* (see I, §C), the S-operations generate what are known as λ-operations. If $\mathrm{A} =$ *G-sets* (see I, §B), the S-operations will be referred to as β-operations.

For $T \in \mathrm{A}^{\mathrm{G}}$, let $\langle T \rangle$ denote the subring of $K_0(\mathrm{A}^{\mathrm{G}})$ generated by $\{[\phi(T)] \mid \phi$ an S-operation$\}$. If there is a functor Hom as in Lemma II.11, then Theorem II.15 says that every element in $\langle T \rangle$ is a finite sum $\Sigma_\alpha q_\alpha [\phi_\alpha(T)]$, where $q_\alpha \in \mathbf{Z}$, ϕ_α an S-operation.

Summarizing, the ring $K_0(\mathrm{A}^{\mathrm{G}})$ and subrings $P(\mathrm{A}^{\mathrm{G}})$ and $\langle T \rangle$ have been constructed. The main theorem and its immediate corollary apply when $G = S_k$ and $T = X_k$:

Main Theorem II.21. *Let* A *have finite coproducts, coequalizers, a "tensor product"* $\perp$, *and an object* $\mathbf{1}$. *Assume there is a functor* Hom *as in Lemma* II.11. *Then for each positive integer* k, $P(\mathrm{A}^{\mathrm{S_k}}) \subset \langle X_k \rangle$.

Corollary II.22. *Same hypothesis as above. Suppose* $P(\mathrm{A}^{\mathrm{S_k}}) = K_0(\mathrm{A}^{\mathrm{S_k}})$. *Then* $\langle X_k \rangle = K_0(\mathrm{A}^{\mathrm{S_k}})$.

Lemma II.23. *Same hypothesis as above. Let* $H \subset S_n$ *and* $W_n = \mathrm{Ind}_H^{S_n} \mathbf{1}$. *Then*

$$\phi_{W_n}(X_k) = \coprod_\gamma \mathrm{Ind}_{H_\gamma}^{S_k} \mathbf{1},$$

for some collection of subgroups H_γ *of* S_k. *Here,* $\gamma_1 \neq \gamma_2$ *need not imply* $H_{\gamma_1} \neq H_{\gamma_2}$.

This lemma does not imply $\langle X_k \rangle \subset P(\mathrm{A}^{\mathrm{S_k}})$, since $\langle X_k \rangle$ is obtained from all S-operations ϕ_{W_n}, and if $P(\mathrm{A}^{\mathrm{S_n}}) \neq K_0(\mathrm{A}^{\mathrm{S_n}})$, W_n need not be a linear combination of objects $\mathrm{Ind}_H^{S_n} \mathbf{1}$.

Proof of lemma. By Theorem II.12, $\phi_{W_n}(X_k) = (\mathrm{Res}_H^{S_n}(X_k^{\perp n}))_H$. Since

$$X_K^{\perp n} = \left(\coprod_{j=1}^{k} \mathbf{1}_j \right)^{\perp n} \approx \coprod_{\substack{(j_1, \cdots, j_n) \\ 1 < j_i < k}} (\mathbf{1}_{j_1} \perp \cdots \perp \mathbf{1}_{j_n})$$

$$\approx \coprod_{\substack{(j_1, \cdots, j_n) \\ 1 < j_i < k}} \mathbf{1}_{(j_1, \cdots, j_n)},$$

we have

$$\phi_{W_n}(X_k) = \left(\mathrm{Res}_H^{S_n} \left(\coprod_{1 < j_i < k} \mathbf{1}_{(j_1, \cdots, j_n)} \right) \right)_H. \tag{1}$$

$\sigma \in S_n$ acts on $\amalg 1_{(j_1,\dots,j_n)}$ by

$$\sigma\colon\ 1_{(j_1,\dots,j_n)} \xrightarrow{1_1} 1_{(j_{\sigma^{-1}(1)},\dots,j_{\sigma^{-1}(n)})} \hookrightarrow \coprod 1_{(j_1,\dots,j_n)},$$

and $g \in S_k$ by

$$g\colon\ 1_{(j_1,\dots,j_n)} \xrightarrow{1_1} 1_{(g(j_1),\dots,g(j_n))} \hookrightarrow \coprod 1_{(j_1,\dots,j_n)};$$

moreover, $\sigma g = g\sigma\colon \amalg 1_{(j_1,\dots,j_n)} \to \amalg 1_{(j_1,\dots,j_n)}$.

Let $J = \{(j_1, \cdots, j_n) \mid 1 \leq j_i \leq k\}$. By the above, $H \subset S_n$ acts on the set J. For $j \in J$, let $\bar{j}$ denote the orbit Hj. Using equation (1), it is not hard to show that $\phi_{W_n}(X_k) = \amalg_{H\text{-orbits}\,\bar{j}} 1_{\bar{j}}$, where $g \in S_k$ acts by $g\colon 1_{\bar{j}} \xrightarrow{1_1} 1_{\overline{gj}} \hookrightarrow \amalg 1_{\bar{j}}$. Let

$$\pi\colon \coprod_j 1_j \to \coprod_{\bar{j}} 1_{\bar{j}}$$

be the map induced from $1_j \xrightarrow{1_1} 1_{\bar{j}} \hookrightarrow \amalg_{\bar{j}} 1_{\bar{j}}$. It is straightforward to show that $(\amalg_{\bar{j}} 1_{\bar{j}}, \pi)$ satisfies the universal property defining $(\amalg_{j\in J} 1_j)_H$. Hence as an object in **A**, $\phi_{W_n}(X_k) = \amalg_{\bar{j}} 1_{\bar{j}}$. Since, for $g \in S_k$, $(\pi g)h = \pi hg = \pi g\colon \amalg 1_j \to \amalg 1_{\bar{j}}$ for all $h \in H$, there is a unique map $\amalg 1_{\bar{j}} \xrightarrow{g} \amalg_{\bar{j}} 1_{\bar{j}}$ such that the diagram

$$\begin{tikzcd}
\amalg_j 1_j \arrow[r, "g"] \arrow[d, "\pi"'] & \amalg 1_j \arrow[d, "\pi"] \\
\amalg 1_{\bar{j}} \arrow[r, "g"'] & \amalg 1_{\bar{j}}
\end{tikzcd}$$

commutes, and hence is determined by the commutative diagram:

$$\begin{tikzcd}
1_j \arrow[r] \arrow[d] & 1_{gj} \arrow[d] \\
1_{\bar{j}} \arrow[dr] & 1_{\overline{gj}} \arrow[d, hook] \\
 & \amalg 1_{\bar{j}}
\end{tikzcd}$$

Thus $\phi_{W_n}(X_k) = \amalg_{H\text{-orbits}\,\bar{j}} 1_{\bar{j}}$, and S_k permutes the $1_{\bar{j}}$'s by permuting the H-orbits $\bar{j}$. Therefore,

$$\phi_{W_n}(X_k) \approx \coprod_{\bar{j}} 1_{\bar{j}} \approx \coprod_{S_k\text{-orbits}\,\gamma} \Bigl(\coprod_{\bar{j}\in\gamma} 1_{\bar{j}} \Bigr) \approx \operatorname{Ind}_{H_\gamma}^{S_k} 1_{\bar{j}_0}$$

(by Proposition II.7), where $1_{\bar{j}_0}$ is one of the $1_{\bar{j}}$'s and H_γ is its isotropy group.□

Let $H \subset S_k$ and m be a nonnegative integer. H is said to be divisible by S_m if H is conjugate to $M \times S_m$ (as subgroups of S_k) for some subgroup M of S_{k-m}. Here, "H conjugate to $M \times S_0$" means H is conjugate to some subgroup M of S_k; "H conjugate to $M \times S_k$" means H is conjugate to S_k. Clearly H is always divisible by S_0.

PROOF OF MAIN THEOREM. It is enough to show that if $H \subset S_k$ is divisible by S_m for some m, $0 \leqslant m \leqslant k$, then $\mathrm{Ind}_H^{S_k} 1 \in \langle X_k \rangle$. The proof is by induction (backwards) on m:

(i) If $m = k$, then $H = S_k$ and $\mathrm{Ind}_H^{S_k} 1 = 1_{S_k} = \phi_0(X_k) \in \langle X_k \rangle$.

(ii) Suppose $m < k$ and assume that if $m < m' \leqslant k$, then H divisible by $S_{m'} \Rightarrow \mathrm{Ind}_H^{S_k} 1 \in \langle X_k \rangle$. Let H be divisible by S_m. Then $\mathrm{Ind}_H^{S_k} 1 = \mathrm{Ind}_{M \times S_m}^{S_k} 1$ for some $M \subset S_{k-m}$. Let $W_{k-m} = \mathrm{Ind}_M^{S_{k-m}} 1$. Lemma II.23 gives

$$\phi_{W_{k-m}}(X_k) = \coprod_{S_k\text{-orbits } \gamma} \left(\coprod_{\bar{j} \in \gamma} 1_{\bar{j}} \right) = \coprod_{\gamma} \mathrm{Ind}_{H_\gamma}^{S_k} 1.$$

Recall that $J = \{(j, \cdots, j_{k-m}) | 1 \leqslant j_i \leqslant k\}$ is an S_k-set and an M-set, that S_k permutes the M-orbits $\bar{j}$ of J, and that γ runs through the S_k-orbits of the set of M-orbits of J.

Direct computation shows that the M-orbit $(\overline{1, \cdots, k-m})$, which is in some S_k-orbit γ_0, has isotropy group $H_{\gamma_0} = M \times S_m$. Moreover, $(\overline{j_1, \cdots, j_{k-m}}) \in \gamma_0$ whenever all the j_i's are distinct, since S_k is $(k-m)$-fold transitive. Thus if $(\overline{j_1, \cdots, j_{k-m}}) \in \gamma \neq \gamma_0$, $j_i = j_t$ for some $i \neq t$; hence its isotropy group H_γ is of the form $K \times S_{m'}$, for some $m' > m$ and $K \subset S_{k-m}$. Since H_γ is divisible by $S_{m'}$, for some $m' > m$ if $\gamma \neq \gamma_0$, $\mathrm{Ind}_{H_\gamma}^{S_k} 1 \in \langle X_k \rangle$ for all $\gamma \neq \gamma_0$ by induction hypothesis. Thus

$$\mathrm{Ind}_H^{S_k} 1 = \mathrm{Ind}_{M \times S_m}^{S_k} 1 = \phi_{W_{k-m}}(X_k) - \sum_{\gamma \neq \gamma_0} \mathrm{Ind}_H^{S_k} 1 \in \langle X_k \rangle.$$

The proof is completed by induction.□

III. APPLICATIONS AND OPEN QUESTIONS

A. **Unigeneration of the λ-ring $R(S_k)$.** It is well known that $R(S_k)$ is a free Z-module with basis $\{\mathrm{Ind}_{S_{k_1} \times S_{k_2} \times \cdots S_{k_s}}^{S_k} 1 | k_i \geqslant 1, \Sigma_i k_i = k\}$ [5, Chapter III]. Therefore, $P(\mathrm{A}^{\mathrm{S_k}}) = K_0(\mathrm{A}^{\mathrm{S_k}}) = R(S_k)$, where $\mathrm{A} =$ *finite-dimensional vector spaces over* C. Corollary II.22 now implies $R(S_k) = \langle X_k \rangle$, and Theorem II.15 gives that every element of $R(S_k)$ is a linear combination of $\{[\phi(X_k)] | \phi$ an S-operation$\}$.

Moreover, every S-operation is a linear combination of symmetric power operations:

$$W_n \in R(S_n) \Rightarrow [W_n] = [W'_n] - [W''_n],$$

where

$$W'_n = \sum \alpha_\mu \mathrm{Ind}^{S_n}_{S_{\mu_1}\times\cdots\times S_{\mu_s}} \mathbf{1}, \quad W''_n = \sum \beta_\nu \mathrm{Ind}^{S_n}_{S_{\nu_1}\times\cdots\times S_{\nu_t}} \mathbf{1},$$

with α_μ, β_ν positive integers.

$$[W'_n] = [W_n] + [W''_n] = [W_n \amalg W'_n] \Rightarrow W'_n \cong W_n \amalg W''_n \qquad \text{(See Remark II.17)}$$

$$\Rightarrow \phi_{W'_n}(T) = \phi_{W_n}(T) \amalg \phi_{W''_n}(T) \quad \text{(by Theorem II.9)}$$

$$\Rightarrow [\phi_{W_n}(T)] = [\phi_{W'_n}(T)] \amalg [\phi_{W''_n}(T)].$$

Since

$$\mathrm{Ind}^{S_n}_{S_{n_1}\times S_{n_2}} \mathbf{1} = \mathrm{Ind}^{S_n}_{S_{n_1}\times S_{n_2}} (\mathbf{1}\otimes\mathbf{1}),$$

etc., Theorem II.15 implies $\phi_{W_n}(T)$ is a linear combination of $\{h_{n_1}(T) \otimes h_{n_2}(T) \otimes \cdots \otimes h_{n_s}(T) | n_i \geqslant 0\}$, where $h_0 = \phi_0$, and $h_n = \phi_{W_n}$ for $n > 0$ and $W_n = \mathrm{Ind}^{S_n}_{S_n}\mathbf{1}$. The h_i's are, of course, symmetric power operations (see Example II.13(ii)).

Combining the two paragraphs above, one obtains the result that every element of $R(S_k)$ is a linear combination of $\{h_{n_1}(X_k) \otimes \cdots \otimes h_{n_s}(X_k) | n_i \geqslant 0\}$. Thus $R(S_k)$ is generated by the single element X_k if symmetric powers are included with the standard ring operations. Since λ-operations generate symmetric power operations [2], [5], X_k generates $R(S_k)$ as a λ-ring.

REMARK III.1. Although $R(S_k)$ is unigenerated as a λ-ring, it is not unigenerated as a ring, i.e., $R(S_k) \neq \mathbf{Z}[T]$ for all $T \in R(S_k)$. The first counterexample is $R(S_4)$:

If $R(S_4) = \mathbf{Z}[T]$, then the ring $\mathbf{Z}/2 \otimes_{\mathbf{Z}} R(S_4)$ is unigenerated as a $\mathbf{Z}/2$-module. Since $R(S_4)$ is a free $\mathbf{Z}$-module of rank 5 (see Proposition I.9), $\mathbf{Z}/2 \otimes_{\mathbf{Z}} R(S_4)$ is a free $\mathbf{Z}/2$-module of rank 5. By writing out its multiplication table ($\mathbf{Z}/2 \otimes_{\mathbf{Z}} R(S_4)$ has only 2^5 elements), one can show that no element generates all of $\mathbf{Z}/2 \otimes_{\mathbf{Z}} R(S_4)$.

B. A unigeneration theorem for $B(S_k)$. $B(G)$ is a free $\mathbf{Z}$-module with basis $\{G/H_\alpha\}$, where $\{H_\alpha\}$ = a set of representatives of the conjugacy classes of subgroups of G (Proposition I.5). Clearly, if A = *finite sets*, then $\mathrm{Ind}^G_H \mathbf{1}$ = the G-set G/H (see Proposition II.18). Thus $P(\mathsf{A}^G) = K_0(\mathsf{A}^G) = B(G)$. Hence Corollary II.22 implies that $B(S_k) = \langle X_k \rangle$. Thus S-operations (here called "β-operations") applied to X_k generate all of $B(S_k)$.

REMARK III.2. $B(S_k)$ is not, in general, generated by one element as a ring, since the ring homomorphism $B(G) \longrightarrow R(G)$ defined by $T \longmapsto$ vector space with basis $\{v_t\}_{t\in T}$ (see Examples I.6(ii), (ii′)) is onto if $G = S_k$ [5, Chapter III], and therefore the ring $B(S_4)$ is not unigenerated since $R(S_4)$ is not (see Remark III.1).

REMARK III.3. Applying sums and products of symmetric power operations h_n to X_k does not, in general, give all of $B(S_k)$. $B(S_3)$ is a counterexample:

The nonconjugate subgroups of S_3 are 1, $S_1 \times S_2$, S_3 and A_3 (= the even permutations in S_3). Proposition I.5 now says that $B(S_3)$ is a free **Z**-module with basis $S_3/1$, $S_3/S_1 \times S_2$, S_3/S_3, S_3/A_3. Note that $S_3/S_1 \times S_2 = X_3$ (see Proposition II.19) and $S_3/S_3 = 1$. $B(S_3)$ is now completely described by the following multiplication table, which is obtained easily by direct calculation using Propositions I.2, I.3, and I.4:

	1	S_3/A_3	X_3	$S_3/1$
1	1	S_3/A_3	X_3	$S_3/1$
S_3/A_3	S_3/A_3	$2S_3/A_3$	$S_3/1$	$2S_3/1$
X_3	X_3	$S_3/1$	$S_3/1 + X_3$	$3S_3/1$
$S_3/1$	$S_3/1$	$2S_3/1$	$3S_3/1$	$6S_3/1$

Now suppose that the symmetric power operations h_n applied to X_3 give all of $B(S_3)$. Then, in particular, S_3/A_3 could be expressed as a finite sum

$$\sum a_{i_1\cdots s} h_{n_{i_1}}(X_3) h_{n_{i_2}}(X_3) \cdots h_{n_{i_s}}(X_3), \quad \text{where } a_{i_1\cdots s} \in \mathbf{Z}.$$

From the multiplication table, it is clear that one of the $h_n(X_3)$'s above must be of the form $n_1 1 + n_2 S_3/A_3 + n_3 X_3 + n_4 S_3/1$, with $n_2 \neq 0$. But for all $n \geqslant 0$, $h_n(X_3) = n_1 1 + n_3 X_3 + n_4 S_3/1$ for some $n_i \in \mathbf{Z}$: An element $(x_1, x_2, \cdots, x_n)$ in an S_n-orbit of X_3^n is made up of μ_1 1's, μ_2 2's, μ_3 3's, where $\mu_1 + \mu_2 + \mu_3 = n$, and the 3-tuple (μ_1, μ_2, μ_3) uniquely determines the S_n-orbit. If the μ_i's are all different, then the S_3-orbit of $(X_3^n)_{S_n}$ which contains the S_n-orbit corresponding to (μ_1, μ_2, μ_3) is $S_3/1$. If exactly two of the μ_i's are the same, then the S_3-orbit is $S_3/S_1 \times S_2$. If $\mu_1 = \mu_2 = \mu_3$, then the S_3-orbit is S_3/S_3. Therefore, S_3/A_3 never arises.

Thus $B(S_k)$ is generated by X_k if all the β-operations are used, but is, in general, not generated by X_k if only symmetric power operations are used. Hence β-operations include, but are not the same as, symmetric powers.

C. Some open questions. Since the S-operations in the linear representation theory case are generated by symmetric power operations (see III, §A), which are defined on all of $R(G)$ (see I, §C), S-operations extend to operations on $R(G)$, thus making $R(G)$ a "λ-ring". The unigeneration of $R(S_k)$ can be phrased:

There is an onto "λ-ring homomorphism"

$$\Lambda \longrightarrow R(S_k), \qquad a_1 \longmapsto X_k,$$

where Λ is the "free λ-ring on one generator" $a_1 \in \Lambda$. Hence $R(S_k) \cong \Lambda/I$, for some λ-ideal I. A reasonable description of this λ-ideal, in particular, a canonical set of generators, is unknown.

In the case of permutation representations, i.e., G-sets, the corresponding theory of "β-ring" which would allow extending the β-operations to all of $B(G)$ is not known.

REFERENCES

1. M. Atiyah, *Power operations in K-theory*, Quart. J. Math. Oxford Ser. (2) 17 (1966), 165–193. MR 34 #2004.

2. M. Atiyah and D. O. Tall, *Group representations, λ-rings, and the J-homomorphism*, Topology 8 (1969), 253–297. MR 39 #5702.

3. H. Bass, *Algebraic K-theory*, Benjamin, New York 1968. MR 40 #2736.

4. A. Dress, *Representations of finite groups*. 1. *The Burnside ring* (mimeographed notes, Bielefeld, 1971).

5. D. Knutson, *λ-rings and the representation theory of the symmetric group*, Lecture Notes in Math., vol. 308, Springer-Verlag, Berlin and New York, 1973.

6. B. Mitchell, *Theory of categories*, Academic Press, New York, 1965. MR 34 #2647.

7. N. Saavedra Rivano, *Categories Tannakiennes*, Lecture Notes in Math., vol. 265, Springer-Verlag, Berlin and New York, 1972.

8. J.-P. Serre, *Representations linéaires des groupes finis*, Hermann, Paris, 1967. MR 38 #1190.

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